

Abelian symmetries as the source of the small neutrino-related flavor parameters

Shahar Amitai*

*Department of Particle Physics and Astrophysics,
Weizmann Institute of Science, Rehovot 76100, Israel*

Abstract

There are four neutrino-related flavor parameters that have been measured: The three mixing angles s_{23} , s_{12} , s_{13} , and the ratio of mass-squared differences $r_{23} \equiv \Delta m_{21}^2 / |\Delta m_{31}^2|$. Of these, the first two are order one. On the other hand, s_{13} and r_{23} can be either order-one parameters that are accidentally somewhat small, or they are small for a reason, for example, they vanish in the limit of a symmetry that is broken by small parameters. We show that in the latter case, the Froggatt-Nielsen mechanism could explain the smallness of s_{13} and r_{23} only if some order-one coefficients are as small as the symmetry-breaking parameters. It is thus very unlikely that an Abelian symmetry is responsible for the smallness of s_{13} and r_{23} .

1 Introduction

The flavor parameters in the quark sector – quark masses and CKM mixing angles – exhibit smallness and hierarchy. This structure is special enough that it is unlikely to be accidental. A leading candidate to explain these features is an approximate symmetry. The smallness is explained by the vanishing of all the small parameters in the symmetry limit. The hierarchy is explained by different powers of the small symmetry-breaking parameter(s) entering the various Yukawa couplings. The simplest successful attempt to explain the smallness and hierarchy in the quark flavor parameters is provided by the Froggatt-Nielsen mechanism [1] which employs a $U(1)$ symmetry. It predicts several relations among the flavor parameters that are all consistent with the measured values [2].

The situation in the lepton sector is less clear. The recent measurements of $|U_{e3}|$ [3, 4, 5, 6, 7] have extended the list of measured lepton flavor parameters to eight: Three mixing angles, two neutrino mass-squared differences, and three charged lepton masses. Of these, there are four dimensionless parameters that seem to be small (See [8, 9, 10] for the neutrino parameters, and [11] for the charged lepton masses):

$$s_{13} \equiv \sin \theta_{13} = 0.15 , \quad (1)$$

$$r_{23} \equiv \Delta m_{21}^2 / |\Delta m_{31}^2| = 0.031 , \quad (2)$$

$$m_\mu / m_\tau = 0.059 , \quad (3)$$

$$m_e / m_\mu = 0.0048 . \quad (4)$$

*shahar.amitai@weizmann.ac.il

We quote here only the central values, because the experimental errors are small enough that they cannot change our analysis. The two other mixing angles, $s_{23} \equiv \sin \theta_{23}$ and $s_{12} \equiv \sin \theta_{12}$, are not small. One may wonder why we do not list m_τ/v (with $v \simeq 246$ GeV the vacuum expectation value (VEV) of the Standard Model (SM) Higgs) as a small flavor parameter. The reason is that the smallness of m_τ/v may be the result of a small VEV v_ℓ of the scalar that couples to the charged leptons, $v_\ell \sim m_\tau$. Of course, if, instead, it is a result of a small tau-Yukawa, it should be added to the above list. Similarly, we do not consider each neutrino mass-squared difference as a small flavor parameter, because the overall scale of the neutrino masses is likely to be related to the seesaw mechanism rather than to flavor physics; We do not know whether the neutrino masses are small compared to the scale of $v^2/\Lambda_{\text{seesaw}}$ or not.

The issue of flavor model building depends crucially on how one interprets the data. In the context of an approximate symmetry, one has to decide which observables are truly small and therefore must vanish in the symmetry limit, and which ones are order-one parameters (which may accidentally assume somewhat small values), not suppressed by powers of symmetry-breaking parameters. Our own judgement is that the charged lepton mass ratios, m_e/m_μ (4) and m_μ/m_τ (3), are small, and that therefore m_e and m_μ should vanish in the symmetry limit. On the other hand, the value of s_{13} (1) is large enough that both options – accidental smallness and parametric suppression – are possible.

As concerns r_{23} (2), things depend on whether neutrino masses have normal hierarchy, inverted hierarchy, or quasi-degeneracy. For normal hierarchy, the relevant flavor parameter is $m_2/m_3 \approx \sqrt{r_{23}} \simeq 0.18$, comparable to s_{13} and possibly just accidentally small. With inverted hierarchy, the relevant flavor parameter is $\Delta m_{21}^2/m_2^2 \simeq r_{23}$, close to the charged lepton mass ratios and probably truly small. In the case of quasi-degeneracy, $\Delta m_{21}^2/m_2^2 \ll r_{23}$ and should definitely be parametrically suppressed. In all cases, models in which the smallness of s_{13} is explained from symmetry reasons, but the smallness of r_{23} is put by hand, make little sense to us.

Our aim in this work is to answer a simple question: Can an approximate Abelian symmetry account for the smallness of all four lepton flavor parameters of Eqs. (1)-(4)? If the answer is in the affirmative, then it is interesting to identify the relevant class of models and to find whether they lead to further predictions. If not, then we would be led to consider two possibilities: Either the explanation to smallness and hierarchy does not lie in the framework of approximate Abelian symmetries, or the somewhat small values of s_{13} and $\sqrt{r_{23}}$ are just accidentally so.

2 The framework

The leptonic mass terms take the following form:

$$\mathcal{L}_{\text{mass}}^{\text{leptons}} = \nu_i (M_\nu)_{ij} \nu_j + \overline{\ell}_{Li} (M_E)_{ij} \ell_{Rj} + \text{h.c.} \quad , \quad (5)$$

where i and j are flavor indices, M_ν is symmetric, and we take the neutrinos to be of Majorana type. The two mass matrices can be diagonalized as follows:

$$U_\nu^T M_\nu U_\nu = D_\nu \quad , \quad U_\ell^\dagger M_E M_E^\dagger U_\ell = D_e^2 \quad , \quad (6)$$

where $D_\nu = \text{diag}(m_1, m_2, m_3)$ and $D_e = \text{diag}(m_e, m_\mu, m_\tau)$. The leptonic mixing matrix is then given by

$$U = U_\ell^\dagger U_\nu \quad . \quad (7)$$

We assume that the entries of M_ν and M_E are subject to selection rules that arise from an approximate continuous Abelian symmetry. We denote the charges of the $SU(2)$ -doublet leptons under this symmetry by H_i^L , and those of the charged $SU(2)$ -singlet leptons by H_i^R . As long as we are not interested in the structure of the scalar potential, we can always choose the charges of the relevant Higgs fields to be zero. In the symmetry limit,

$$(M_\nu)_{ij} = 0 \quad \text{if} \quad H_i^L + H_j^L \neq 0 \quad (8)$$

(and order one otherwise). Thus, in the symmetry limit, a diagonal entry would be different from zero only if the corresponding field carries charge zero, while an off-diagonal entry would be different from zero only if the sum of charges of the corresponding fields is zero. This simple observation implies that, in the symmetry limit, there are seventeen possible structures for M_ν : one where all entries are allowed (all $H_i^L = 0$), one where all entries vanish, and five classes of other forms, each standing for three matrices that are related by flavor permutations. For each of the seventeen forms there is a large set of possible M_E where, in the symmetry limit,

$$(M_E)_{ij} = 0 \quad \text{if} \quad H_i^L - H_j^R \neq 0 \quad (9)$$

(and order one otherwise). Let us clarify that we use the term “order one” to mean that these parameters are not suppressed by the small symmetry-breaking parameters, though their exact numerical value is not determined by the symmetry.

3 Explaining $r_{23} \ll s_{13} \ll 1$

One of the main challenges in lepton flavor model building, when assuming $s_{13} = 0$, is to find a symmetry that allows two mixing angles of order one, but makes the third mixing angle vanish. As a first stage, we examine all types of symmetries specified above, and check which ones produce a single vanishing mixing angle in the symmetry limit. In other words, we examine the 17 forms of M_ν and, for each of them, all the relevant forms of M_E . For each type of symmetry, we find U_ν and U_ℓ by diagonalizing the most general M_ν and $M_E M_E^\dagger$, respectively, and then obtain the mixing matrix U . Note that permuting rows or columns of U corresponds to reordering the charged lepton or neutrino masses. Therefore, if we obtain a zero entry in U , it can always be moved to be U_{e3} . In addition, whenever we have a degenerate subspace (in terms of the neutrino or charged lepton masses) we have the freedom to rotate it by an arbitrary angle. In these cases, without loss of generality, we choose U_ν or U_ℓ that gives the largest number of vanishing mixing angles¹.

Going through this procedure, we obtain a single vanishing mixing angle for two types of symmetries:

- **Type I:** Symmetries with $H_e^L = +1$, $H_\mu^L = H_\tau^L = -1$, and at least one of the H_i^R charges equal to -1 ². These symmetries give M_E of various forms, but the same $M_E M_E^\dagger$:

$$M_\nu = \begin{pmatrix} & \bullet & \bullet \\ \vdots & & \end{pmatrix}, \quad M_E M_E^\dagger = \begin{pmatrix} \bullet & & \\ & \vdots & \vdots \\ & & \end{pmatrix}, \quad (10)$$

¹Indeed, the degeneracy of the three neutrino masses within the SM ($M_\nu = 0$) is the reason for the absence of lepton mixing within this model.

²This is up to a flavor permutation: $(H_e^L, H_\mu^L, H_\tau^L) = (-1, +1, -1)$ or $(-1, -1, +1)$ also belong to this type.

where the empty entries vanish in the symmetry limit, and the others are of order one. For symmetries with no $H_i^R = +1$, we have also $(M_E M_E^\dagger)_{11} = 0$. All these symmetries lead to $\theta_{13} = 0$ and a maximal θ_{12} . It is easy to reach this conclusion starting from Eq. (10): An appropriate 2 – 3 rotation of M_ν will make $(M_\nu)_{13/31}$ vanish, and then a maximal 1 – 2 rotation is needed to make M_ν diagonal. Obviously, M_E only needs a 2 – 3 rotation to make it diagonal, so overall we get $\theta_{13} = 0$ and $\theta_{12} = \pi/4$ ³.

- **Type II:** Symmetries with vanishing left-handed charges ($H_e^L = H_\mu^L = H_\tau^L = 0$), and with only two non-vanishing right-handed charges ($H_e^R = 0$ or $H_\mu^R = 0$ or $H_\tau^R = 0$). These symmetries give M_ν completely general and M_E with only one non-vanishing column. Since we have a 2×2 degenerate subspace for the charged leptons, we can make one of the mixing angles vanish. The two other mixing angles are unknown.

We next require that the Abelian symmetry imposes $r_{23} = 0$ in the symmetry limit. Notice that the value of r_{23} only depends on M_ν , namely on the left-handed charges. For Type I symmetries we get $m_1 = -m_2$ and $m_3 = 0$, which indeed gives $r_{23} = 0$. For Type II symmetries, the form of M_ν is completely general, so r_{23} is expected to be of order one, and therefore these symmetries do not explain its smallness.

We now move away from the symmetry limit, and check whether the measured values of the small parameters r_{23} and s_{13} can be accounted for by the small symmetry-breaking parameters. Take, for example, the well-known Type I (vector-like) symmetry:

$$H = L_e - L_\mu - L_\tau \quad , \quad (11)$$

where L_i is the corresponding lepton flavor charge:

$$(H_e^L, H_\mu^L, H_\tau^L) = (H_e^R, H_\mu^R, H_\tau^R) = (+1, -1, -1) \quad . \quad (12)$$

If this symmetry is approximate at low energy, we obtain

$$M_\nu = m_\nu \begin{pmatrix} \varepsilon' & 1 & 1 \\ 1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{pmatrix} \quad , \quad M_E = m_e \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon' & 1 & 1 \\ \varepsilon' & 1 & 1 \end{pmatrix} \quad . \quad (13)$$

For each entry in either mass matrix we only specify the parametric suppression, and omit arbitrary order-one coefficients. ε and ε' are spurions, namely small symmetry-breaking parameters of H -charges +2 and –2, respectively. Using these mass matrices, we get first-order corrections to the predictions derived in the symmetry limit:

$$\theta_{13} \sim \varepsilon + \varepsilon' \quad , \quad \frac{\pi}{4} - \theta_{12} \sim \varepsilon + \varepsilon' \quad , \quad r_{23} \sim 4(\varepsilon + \varepsilon') \quad . \quad (14)$$

Note that the neutrino spectrum is of the inverted hierarchy type and, therefore, as explained in the introduction, the relevant related small parameter is indeed r_{23} (and not $\sqrt{r_{23}}$). The current experimental ranges of these parameters imply

$$\theta_{13} \sim 0.15 \quad , \quad \frac{\pi}{4} - \theta_{12} \sim 0.20 \quad , \quad r_{23} \sim 0.03 \quad . \quad (15)$$

³We are using the common convention $U = R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12})$.

We see a mismatch between the values of the two mixing angles and the value of r_{23} . While θ_{13} and θ_{12} require symmetry-breaking parameters of the order $0.1 - 0.2$, the smallness of r_{23} requires the same symmetry-breaking parameters to be of order 0.01 (Notice the factor of 4 in Eq. (14)). If we want all three parameters to assume their experimental values, we need accidental numbers of order one to be as small as the small symmetry-breaking parameters (or as large as the inverse of the small breaking parameters). Under such circumstances, it is fair to conclude that the symmetry (11) fails to give a natural explanation for the small parameters θ_{13} and r_{23} ⁴.

Our statement that the pattern of the first-order corrections to the symmetry limit is inconsistent with phenomenology was based on the specific example of the symmetry (11). Can other Type I symmetries overcome the problem? The form of M_ν (13) is common to all these symmetries, so the parametric suppression of r_{23} is the same. The various Type I symmetries may differ, however, in the first-order form of $M_E M_E^\dagger$ (The zeroth-order form (10) is the same). In particular, first-order corrections to the mixing angles coming from U_ℓ can be different, or even absent. But since M_ν leads to first-order corrections of its own, the problem appears for all these symmetries. We learn that all Type I symmetries fail to explain the fact that $r_{23} \sim s_{13}^2$.

Our assumption that all vanishing mixing angles in the symmetry limit are small when the symmetry is broken does not always hold. In the case where all entries of M_ν vanish ($H_i^L \neq 0$ and $H_i^L + H_j^L \neq 0$ for all i, j), order-one mixing angles can be obtained from the leading order of the symmetry-breaking parameter. But if all left-handed charges are of the same sign, the leading-order structure is one of the 17 we have analyzed. The form of M_E comes from the corresponding set of possible M_E , and therefore these symmetries are included in our analysis. In the case where all entries of M_E vanish ($H_i^L - H_j^R \neq 0$ for all i, j), we give a similar argument: Given a particular structure of M_ν , the set of entries in M_E that are the same order of the symmetry-breaking parameter always form a structure that has been analyzed. Therefore, assuming a single spurion, we get no new leading-order structures for M_E .

The additional requirement that the symmetry explains the smallness of at least two of the charged lepton masses, Eqs. (3) and (4), further constrains the list of possible symmetries. In particular, it dictates charges of the $SU(2)$ -singlet charged leptons. For example, for the Type I symmetry with $(H_e^L, H_\mu^L, H_\tau^L) = (+1, -1, -1)$ to generate two massless charged leptons (in the symmetry limit), one (and only one) of the right-handed charges H_i^R must equal -1 , while the other two charges must not equal $+1$. These extra constraints, however, have no effect on our conclusion of the failure of these symmetries to account for $r_{23} \ll s_{13} \ll 1$.

So far, we studied $U(1)$ symmetries. One may wonder whether more elaborate continuous symmetries or, more specifically, $[U(1)]^n$ symmetries, can lead to different conclusions. When we combine two symmetries, the allowed entries of the mass matrices (in the symmetry limit) are the intersection of the two sets of allowed entries. Since a larger number of zero entries can only reduce the number of non-vanishing mixing angles, there is no need to consider any $U(1)$ factor that leads to more than a single vanishing mixing angle on its own. We should only consider $U(1)$ factors that lead to one vanishing mixing angle or none. We have listed above two types of $U(1)$ symmetries that lead to a single vanishing mixing angle (Type I and Type II). When considering $U(1)$ factors that lead to no vanishing mixing angle, we are led to define yet another class of symmetries:

- **Type III:** This class includes the three Abelian symmetries L_i^R , defined by having the corresponding $H_i^R \neq 0$ and all other charges zero. By themselves, Type III symmetries give M_ν completely general and M_E with one vanishing column.

⁴For previous analyses of the symmetry $L_e - L_\mu - L_\tau$, see [12, 13]

Any combination of Type II and/or Type III symmetries leaves M_ν completely general, and therefore fails to account for $r_{23} \ll 1$. To account for $r_{23} \ll 1$, Type I symmetries must be involved. Adding Type II or Type III symmetry to Type I symmetry sets columns of M_E to zero. In some cases, this action adds another vanishing mixing angle (in the symmetry limit), and therefore these particular combinations do not work. In other cases, this action changes the symmetry-limit form of M_E to another Type I form. Yet, these combinations do not change the form of M_ν , so r_{23} gets first-order corrections. Therefore, these combinations still fail to account for $r_{23} \ll \theta_{13} \ll 1$.

We are left with the option of combining two Type I symmetries (Having more than two $U(1)$'s does not change the argument below). There are two possible scenarios here. The first scenario involves two $U(1)$'s with different left-handed charges (so each of these $U(1)$ factors by itself gives a different form for M_ν). In the symmetry limit, M_ν has only two allowed entries, while $M_E M_E^\dagger$ is diagonal. Consequently, only one non-vanishing mixing angle is generated. Thus, this scenario is excluded. The second scenario involves two $U(1)$'s with the same left-handed charges (same form for M_ν). In this case, M_ν only gives second-order corrections to the mixing angles and to r_{23} and M_E only gives second-order corrections to the mixing angles. Consequently, the parametric suppression of θ_{13} and r_{23} is still of the same order and the problem remains.

The bottom line is that no low-energy effective continuous Abelian symmetry can give a natural explanation for the small parameters θ_{13} and r_{23} .

4 Discrete Abelian symmetries

In the case of a discrete Abelian symmetry Z_n , we have, without loss of generality, $H_i^{L,R} \in \mathbb{Z}_n$. In the symmetry limit,

$$(M_\nu)_{ij} = 0 \quad \text{if} \quad H_i^L + H_j^L \neq 0 \pmod{n} \quad (16)$$

(and order one otherwise). Since a discrete transformation is a particular case of a continuous one, all matrix structures specified in section 1 for a continuous symmetry can be a result of a discrete symmetry too. A discrete symmetry, though, can produce two additional structures for M_ν (times three flavor permutations) coming from the fact that a diagonal entry can also be different from zero if the corresponding field carries charge $n/2$. For each of these two additional forms there is a large set of possible M_E that, in the symmetry limit, satisfy eq. (9)⁵. Including these extra structures in our analysis, we get no additional types of symmetries that lead to one vanishing mixing angle, or none, in the symmetry limit. We conclude that no low-energy effective Abelian symmetry (continuous, discrete, or a combination of the two) can give a natural explanation for the small parameters θ_{13} and r_{23} .

5 Seesaw mechanism

So far we have assumed the selection rules that come from an approximate Abelian symmetry apply directly to the low-energy effective Lagrangian and, in particular, to the light neutrino mass matrix. In this section we assume that the Majorana mass terms come from a seesaw mechanism and consider the case that the symmetry acts on the full high energy theory, which includes the $SU(2)$ -singlet neutrinos. In such a scenario, the effective neutrino mass matrix M_ν is not necessarily

⁵Choosing $H_i^{L,R} \in \mathbb{Z}_n$ like we did, the only option for the subtraction of two charges to be zero \pmod{n} is zero.

the most general matrix allowed by the approximate symmetry [13]. M_ν would be the most general matrix allowed by the approximate symmetry when all singlet neutrino charges have the same sign, and there is no more than one spurion breaking the symmetry [14, 15]. In terms of the form of M_ν , these cases are equivalent to a low-energy effective theory, and therefore all the previous results hold.

6 Neutrino flavor anarchy

Our statement that the Froggatt-Nielsen mechanism cannot account for the measured values of s_{13} and r_{23} depends crucially on our interpretation of the data as implying that these parameters are parametrically suppressed. The other possible interpretation, that these (or, more precisely, s_{13} and $\sqrt{r_{23}}$) are order-one parameters that are accidentally suppressed by order-one coefficients, is consistent with a $U(1)$ symmetry that suppresses the charged lepton masses, but leaves all neutrino-related parameters unsuppressed. Such a scenario, known as “Anarchy” [16, 17], requires simply $H_e^L = H_\mu^L = H_\tau^L$. The hierarchy in the charged lepton masses is explained by, for example, $H_e^R > H_\mu^R > H_\tau^R \geq 0$.

7 Conclusions

We examined all possible Abelian symmetries that dictate selection rules to the lepton mass matrices, with the aim of explaining the four small parameters of the lepton sector in a natural way. We focussed on symmetries that predict $\theta_{13} = 0$ and $r_{23} \equiv \Delta m_{21}^2 / |\Delta m_{31}^2| = 0$ in the symmetry limit. We found that this symmetries lead to inverted hierarchy and, in the symmetry limit, to maximal $1 - 2$ mixing. We further found that all these models predict r_{23} , θ_{13} and $\pi/4 - \theta_{12}$ of the same parametric suppression, while in reality r_{23} is much smaller than θ_{13} and than $\pi/4 - \theta_{12}$. This situation can only be fixed at the price of introducing order-one parameters that are accidentally as small as the symmetry-breaking parameters. Under such circumstances, it is hard to see the motivation to think that these models apply in Nature.

During recent years, lepton flavor models were evaluated by numerically estimating their “success rate” to produce small enough values for s_{13} and r_{23} . In particular, models with neutrino flavor anarchy were evaluated [16, 18], and were also compared to models that enforce neutrino hierarchy by using the Froggatt-Nielsen mechanism [19, 20]. The success rate of producing both s_{13} and r_{23} smaller than their current measured values is of order 1% at the most. Our results explain the difficulty in achieving a higher success rate.

Our conclusion leaves open two different possibilities within the assumption that the structure in the flavor parameters is the consequence of an approximate symmetry:

- Approximate $U(1)$ symmetries (the Froggatt-Nielsen mechanism) are at work, but the interpretation of $\sqrt{r_{23}}$ and $|U_{e3}|$ as small parameters is incorrect. In other words, there is neutrino flavor anarchy.
- The symmetry that dictates the structure of the flavor parameters is non-Abelian.

Acknowledgments

I thank Yossi Nir for useful discussions and assistance with the draft. I thank Efrat Gerchkovitz and Tali Vaknin for many useful discussions. The research of SA is supported by the Israel Science Foundation.

Appendix A M_ν with $|U_{e3}| = 0$

For the purposes of this appendix, we work in the flavor basis (where M_E is diagonal). In this basis, the mixing matrix is made of the three eigenvectors \vec{U} of M_ν :

$$M_\nu \vec{U} = \lambda \vec{U} \quad . \quad (17)$$

Having $\theta_{13} = 0$ (or $|U_{e3}| = 0$) means that one of the eigenvectors is of the form

$$\vec{U} = \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix} \quad . \quad (18)$$

Eq. (17) provides three relations among the entries of M_ν . Since we introduced two new variables - a and λ , we should expect to get one condition on the entries that is independent of a and λ . Indeed, solving the three equations we get the condition:

$$m_{33} = m_{22} + m_{32} \left(\frac{m_{31}}{m_{21}} - \frac{m_{21}}{m_{31}} \right) \quad . \quad (19)$$

We have found a way to identify mass matrices that lead to $\theta_{13} = 0$. Eq. (19) is satisfied by all these matrices, and only by them⁶. It is indeed satisfied by the mass matrix in a popular $A4$ -based model [21] and also by the mass matrix resulting from the $L_e - L_\mu - L_\tau$ symmetry discussed above:

$$M_\nu^{(A4)} \sim \begin{pmatrix} A + 2B & -B & -B \\ -B & 2B & A - B \\ -B & A - B & 2B \end{pmatrix} \quad , \quad M_\nu^{(L_e - L_\mu - L_\tau)} \sim \begin{pmatrix} 0 & A & B \\ A & 0 & 0 \\ B & 0 & 0 \end{pmatrix} \quad . \quad (20)$$

We can express the constraint of Eq. (19) in a more convenient manner. For example, the matrices

$$M_\nu^{(1)} = \begin{pmatrix} A & B & B \sin \theta \\ B & C & D \sin \theta \\ B \sin \theta & D \sin \theta & C - D \cos \theta^2 \end{pmatrix} \quad , \quad M_\nu^{(2)} = \begin{pmatrix} A & B & B / \sin \theta \\ B & C & D \sin \theta \\ B / \sin \theta & D \sin \theta & C + D \cos \theta^2 \end{pmatrix}$$

are the most general matrices that satisfy eq. (19). Indeed, they have five free parameters instead of six. We can use one of them, depending on which entry is bigger: m_{21} or m_{31} ⁷.

⁶Notice that eq. (19) cannot be satisfied when $m_{21} = 0$ or $m_{31} = 0$. In this case, the solution for the three equations is $m_{21} = m_{31} = 0$. This means that at least two mixing angles vanish, so it is not an interesting scenario.

⁷In terms of the three mixing angles, we get, for $M_\nu^{(1)}$, $\sin \theta = \tan \theta_{23}$, or, for $M_\nu^{(2)}$, $\sin \theta = \cot \theta_{23}$. Indeed, when $\sin \theta = 1$ we get the most general mass matrix that corresponds to $\theta_{13} = 0$ and a maximal θ_{23} .

References

- [1] C. Froggatt and H. B. Nielsen, “Hierarchy of Quark Masses, Cabibbo Angles and CP Violation,” *Nucl.Phys.* **B147** (1979) 277.
- [2] M. Leurer, Y. Nir, and N. Seiberg, “Mass matrix models,” *Nucl.Phys.* **B398** (1993) 319–342, [arXiv:hep-ph/9212278](#) [hep-ph].
- [3] **DAYA-BAY** Collaboration, F. An *et al.*, “Observation of electron-antineutrino disappearance at Daya Bay,” *Phys.Rev.Lett.* **108** (2012) 171803, [arXiv:1203.1669](#) [hep-ex].
- [4] **RENO** Collaboration, J. Ahn *et al.*, “Observation of Reactor Electron Antineutrino Disappearance in the RENO Experiment,” *Phys.Rev.Lett.* **108** (2012) 191802, [arXiv:1204.0626](#) [hep-ex].
- [5] **DOUBLE-CHOOZ** Collaboration, Y. Abe *et al.*, “Indication for the disappearance of reactor electron antineutrinos in the Double Chooz experiment,” *Phys.Rev.Lett.* **108** (2012) 131801, [arXiv:1112.6353](#) [hep-ex].
- [6] **T2K** Collaboration, K. Abe *et al.*, “Indication of Electron Neutrino Appearance from an Accelerator-produced Off-axis Muon Neutrino Beam,” *Phys.Rev.Lett.* **107** (2011) 041801, [arXiv:1106.2822](#) [hep-ex].
- [7] **MINOS** Collaboration, P. Adamson *et al.*, “Improved search for muon-neutrino to electron-neutrino oscillations in MINOS,” *Phys.Rev.Lett.* **107** (2011) 181802, [arXiv:1108.0015](#) [hep-ex].
- [8] G. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo, *et al.*, “Global analysis of neutrino masses, mixings and phases: entering the era of leptonic CP violation searches,” *Phys.Rev.* **D86** (2012) 013012, [arXiv:1205.5254](#) [hep-ph].
- [9] D. Forero, M. Tortola, and J. Valle, “Global status of neutrino oscillation parameters after Neutrino-2012,” [arXiv:1205.4018](#) [hep-ph].
- [10] M. Gonzalez-Garcia, M. Maltoni, J. Salvado, and T. Schwetz, “Global fit to three neutrino mixing: critical look at present precision,” [arXiv:1209.3023](#) [hep-ph].
- [11] **Particle Data Group** Collaboration, J. Beringer *et al.*, “Review of Particle Physics (RPP),” *Phys.Rev.* **D86** (2012) 010001.
- [12] Y. Nir, “PseudoDirac solar neutrinos,” *JHEP* **0006** (2000) 039, [arXiv:hep-ph/0002168](#) [hep-ph].
- [13] R. Barbieri, L. Hall, D. Tucker-Smith, A. Strumia, and N. Weiner, “Oscillations of solar and atmospheric neutrinos,” *JHEP* **9812** (1998) 017, [arXiv:hep-ph/9807235](#) [hep-ph].
- [14] Y. Grossman and Y. Nir, “Lepton mass matrix models,” *Nucl.Phys.* **B448** (1995) 30–50, [arXiv:hep-ph/9502418](#) [hep-ph].

- [15] G. Altarelli and F. Feruglio, “Models of neutrino masses and mixings,” *New J.Phys.* **6** (2004) 106, [arXiv:hep-ph/0405048](#) [hep-ph].
- [16] L. J. Hall, H. Murayama, and N. Weiner, “Neutrino mass anarchy,” *Phys.Rev.Lett.* **84** (2000) 2572–2575, [arXiv:hep-ph/9911341](#) [hep-ph].
- [17] N. Haba and H. Murayama, “Anarchy and hierarchy,” *Phys.Rev.* **D63** (2001) 053010, [arXiv:hep-ph/0009174](#) [hep-ph].
- [18] A. de Gouvea and H. Murayama, “Neutrino Mixing Anarchy: Alive and Kicking,” [arXiv:1204.1249](#) [hep-ph].
- [19] G. Altarelli, F. Feruglio, and I. Masina, “Models of neutrino masses: Anarchy versus hierarchy,” *JHEP* **0301** (2003) 035, [arXiv:hep-ph/0210342](#) [hep-ph].
- [20] G. Altarelli, F. Feruglio, I. Masina, and L. Merlo, “Repressing Anarchy in Neutrino Mass Textures,” [arXiv:1207.0587](#) [hep-ph].
- [21] G. Altarelli and F. Feruglio, “Tri-bimaximal neutrino mixing, $A(4)$ and the modular symmetry,” *Nucl.Phys.* **B741** (2006) 215–235, [arXiv:hep-ph/0512103](#) [hep-ph].